Donaldson-Thomas invariants and resurgence

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Introduction: non-perturbative partition functions and DT invariants

Let X be a (possibly non-compact) Calabi-Yau threefold.

- Can the DT invariants of X be encoded in a geometric structure?

  Gaiotto-Moore-Neizke, Alexandrov-Persson-Pioline, Joyce, Kontsevich-Soibelman, TB, TB-Strachan, ...

- Is the genus expansion in the topological string free energy

$$\mathcal{F}(x,\lambda) = \sum_{g\geq 0}\Big(\sum_{\beta\in H_2(X,\mathbb{Z})} GW_X(\beta,g)e^{2\pi i\omega_{\mathbb{C}}\cdot\beta}\Big)\lambda^{2g-2},$$

  just a formal expansion? Or can it be given a non-perturbative meaning?

  Pasquetti-Schiappa, Grassi-Hatsuda-Mariño, Coman-Longhi-Pomoni-Teschner, Alim-Hollands-Saha-Tulli-Teschner, Grassi-Hao-Neitzke, ...

We will focus on the case when X is the resolved conifold.

1. Coherent sheaves on the resolved conifold

Coherent sheaves on $\mathbb{P}^1$

Let $\mathcal{A} = \mathrm{Coh}(\mathbb{P}^1)$ be the abelian category of coherent sheaves on $\mathbb{P}^1$.

The indecomposable objects of $\mathcal{A}$ are

(i) line bundles $\mathcal{O}_{\mathbb{P}^1}(n)$ for some $n \in \mathbb{Z}$;

(ii) length $k \geq 1$ thickenings $\mathcal{O}_{kx}$ of points $x \in \mathbb{P}^1$.

The Grothendieck group of $\mathcal{A}$ is defined to be

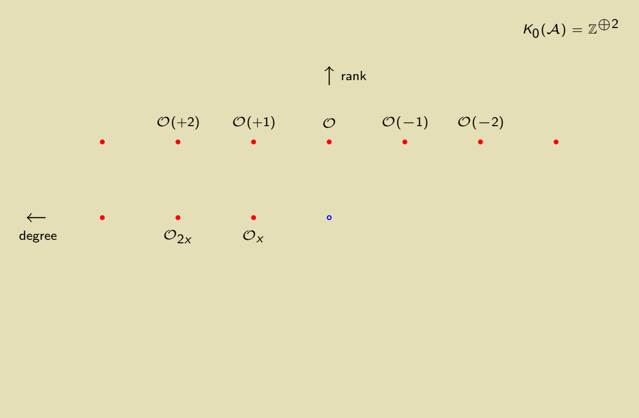
$$K_0(\mathcal{A}) = \bigoplus_{E \in \mathcal{A}/\cong} \mathbb{Z} \cdot [E] \Big/ \left(\begin{array}{c} 0 \to E_1 \to E_2 \to E_3 \to 0 \\ \Longrightarrow [E_2] = [E_1] + [E_3] \end{array} \right).$$

Sending sheaves to their rank and degree defines an isomorphism

$$(r, d)\colon K_0(\mathcal{A}) \longrightarrow \mathbb{Z}^{\oplus 2}.$$

Equivalently $K_0(\mathcal{A}) = \mathbb{Z}\beta \oplus \mathbb{Z}\delta$ is freely generated by $\beta = [\mathcal{O}_{\mathbb{P}^1}]$ and $\delta = [\mathcal{O}_x]$.

Indecomposable objects of $\mathcal{A} = \mathrm{Coh}(\mathbb{P}^1)$



The derived category of $\mathbb{P}^1$

Introduce the bounded derived category $\mathcal{D} = D^b(\mathcal{A})$.

The objects are cochain complexes in $\mathcal{A} = \mathrm{Coh}(\mathbb{P}^1)$ up to quasi-isomorphism.

There is an obvious embedding $\mathcal{A} \hookrightarrow \mathcal{D}$.

The category $\mathcal{D}$ is triangulated: there are shift functors $[k] \colon \mathcal{D} \to \mathcal{D}$ and exact triangles.

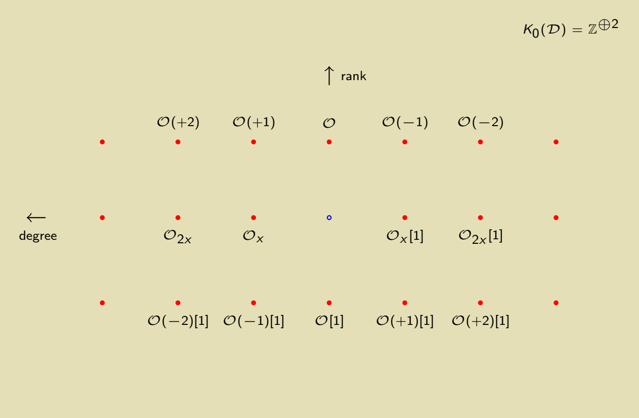
Since $\dim_{\mathbb{C}}(\mathbb{P}^1) = 1$ every $E \in \mathcal{D}$ satisfies $E \cong \bigoplus_{i \in \mathbb{Z}} H^i(E)[-i]$.

So every indecomposable object of $\mathcal{D}$ is a shift of an object from $\mathcal{A}$.

We define the Grothendieck group $K_0(\mathcal{D})$ using triangles instead of exact sequences.

Then $K_0(\mathcal{D}) = K_0(\mathcal{A})$ and $[E[d]] = (-1)^d [E] \in K_0(\mathcal{D})$.

Indecomposable objects of $\mathcal{D} = \mathcal{D}^b\,\mathrm{Coh}(\mathbb{P}^1)$ up to double shift [2]



Stability conditions and DT invariants

Let $\mathcal{D}$ be a triangulated category, e.g. $\mathcal{D} = \mathcal{D}^b \operatorname{Coh}(X)$.

In general, to form nice moduli spaces we need to choose a stability condition.

We first take a "charge lattice" $\Gamma = \mathbb{Z}^{\oplus n}$ with a map $\mathrm{ch}\colon K_0(\mathcal{D}) \to \Gamma$.

A stability condition $\sigma = (Z, \mathcal{P})$ on $\mathcal{D}$ then consists of

- a group homomorphism $Z\colon \Gamma \to \mathbb{C}$ called the central charge,
- a subcategory $\mathcal{P}(\phi) \subset \mathcal{D}$ for each phase $\phi \in \mathbb{R}$ whose objects are called semistable,

together satisfying a short list of axioms.

The set of all stability conditions on $\mathcal{D}$ forms a complex manifold.

If $\mathcal{D}$ has the CY_3 property (and σ is "nice"), we can also define DT invariants

$$\Omega_\sigma(\gamma) \in \mathbb{Q}, \qquad \gamma \in \Gamma,$$

which are virtual Euler characteristics of moduli spaces of σ-semistable objects.

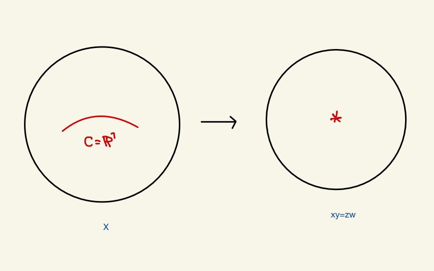
The resolved conifold

Let X to be the total space of the rank 2 bundle $\mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2}$ on $\mathbb{P}^1$.

Then X is a non-compact Calabi-Yau: there is a non-vanishing section of ω_X.

There is a single compact curve $C = \mathbb{P}^1$ in X given by the zero-section.

Contracting C defines a crepant resolution of the affine variety $(xy - zw) \subset \mathbb{C}^4$.



Stability conditions on the resolved conifold

Let $\mathcal{D} \subset \mathcal{D}^b(\mathrm{Coh}(X))$ be the subcategory of compactly-supported objects.

There is a group homomorphism $\mathrm{ch}\colon K_0(\mathcal{D}) \to \Gamma = \mathbb{Z}\beta \oplus \mathbb{Z}\delta$.

Coherent sheaves on $\mathbb{P}^1$ define objects of $\mathcal{D}$ via the inclusion $\mathbb{P}^1 = C \subset X$.

Theorem

Choose $v, w \in \mathbb{C}^*$ with $\mathrm{Im}(v/w) > 0$. Then

(a) *there is a stability condition $\sigma = (Z, \mathcal{P})$ on $\mathcal{D}$, unique up to [2], such that*

- (i) $Z(\beta) = v$ *and* $Z(\delta) = -w$,
- (ii) *the stable objects are* $\{\mathcal{O}_C(n) : n \in \mathbb{Z}\}$ *and* $\{\mathcal{O}_x : x \in X\}$ *and their shifts;*

(b) *the nonzero DT invariants for σ are*

$$\Omega_\sigma(\pm(\beta + n\delta)) = 1 \text{ for } n \in \mathbb{Z}, \qquad \Omega_\sigma(k\delta) = -2 \text{ for } k \in \mathbb{Z} \setminus \{0\}.$$

Central charges in the case $v = i$ and $w = 1$

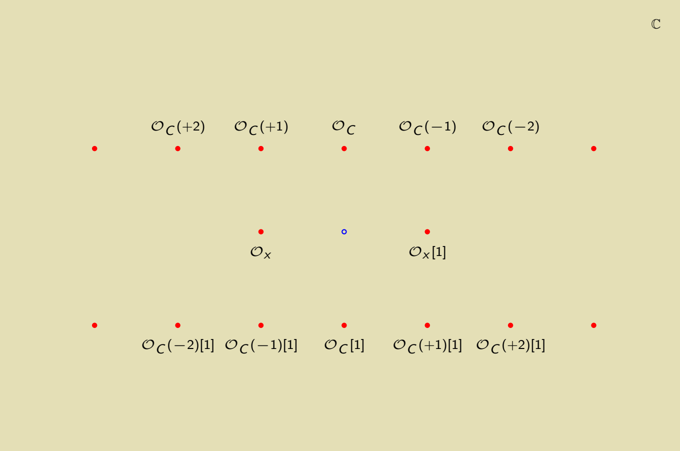


Figure: The images of the stable objects under $Z\colon K_0(\mathcal{D}) \to \mathbb{C}$.

Automorphisms of $(\mathbb{C}^*)^n$ associated to rays

Let $\mathcal{D}$ be a CY_3 triangulated category with a charge lattice $\mathrm{ch}\colon K_0(\mathcal{D}) \to \Gamma \cong \mathbb{Z}^{\oplus n}$.

We assume the skew-symmetric Euler form on $K_0(\mathcal{D})$ descends to

$$\langle -, - \rangle \colon \Gamma \times \Gamma \to \mathbb{Z}.$$

Introduce the torus $\mathbb{T} = \mathrm{Hom}_{\mathbb{Z}}(\Gamma, \mathbb{C}^*) \cong (\mathbb{C}^*)^n$ with character lattice Γ.

It has a Poisson structure: $\{X_{\gamma_1}, X_{\gamma_2}\} = \langle \gamma_1, \gamma_2 \rangle \cdot X_{\gamma_1+\gamma_2}$.

Fix a stability condition $\sigma = (Z, \mathcal{P})$ on $\mathcal{D}$.

To each ray $\ell \subset \mathbb{C}^*$ we can try to associate a Poisson automorphism $\mathbb{S}_\ell$ of $\mathbb{T}$

$$\mathbb{S}(\ell)^*(X_\beta) = X_\beta \cdot \prod_{\gamma \in \Gamma : Z(\gamma) \in \ell} (1 + X_\gamma)^{\Omega_\sigma(\gamma) \cdot \langle \beta, \gamma \rangle}.$$

We need conditions on the growth of the $\Omega_\sigma(\gamma)$ to make analytic sense of this.

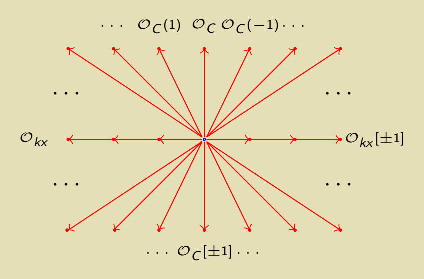
Ray / radar / peacock diagram

In the conifold case $\langle -, - \rangle = 0$, so we replace Γ by

$$\Gamma^\vee \oplus \Gamma = \mathbb{Z}\delta^\vee \oplus \mathbb{Z}\beta^\vee \oplus \mathbb{Z}\beta \oplus \mathbb{Z}\delta.$$

Associated to each ray is a (partially-defined) automorphism of $(\mathbb{C}^*)^4$, e.g.

$$\mathbb{S}_{\ell(\mathcal{O}_C(n))} \colon (X_\delta^\vee, X_\beta^\vee, X_\beta, X_\delta) \mapsto (X_\delta^\vee(1 + X_\beta X_\delta^n)^n, X_\beta^\vee(1 + X_\beta X_\delta^n), X_\beta, X_\delta).$$



2. Borel summation of the genus expansion

Borel summation

Suppose we have a formal complex power series $f(\epsilon) = \sum_{k \geq 0} a_k \epsilon^{k+1}$.

If the coefficients grow like $|a_k| \sim k!$ then the radius of convergence is zero.

Define the Borel transform to be $\hat{f}(\xi) = \sum_{k \geq 0} \frac{a_k}{k!} \xi^k$.

Suppose this sum converges and hence defines an analytic function $\hat{f}(\xi)$ near $\xi = 0$.

Also that $\hat{f}(\xi)$ can be analytically continued along $\mathbb{R}_{>0}$ and doesn't grow too fast.

Then define an analytic function in the half-plane $\mathrm{Re}(\epsilon) > 0$ by the Laplace transform

$$(\mathcal{B}f)(\epsilon) = \int_0^\infty \hat{f}(\xi) e^{-\xi/\epsilon} d\xi.$$

When all this works $(\mathcal{B}f)(\epsilon)$ is called the Borel sum of the series $f(\epsilon)$.

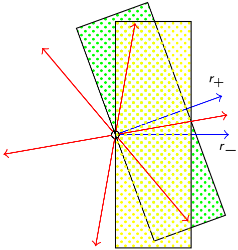
If the series $f(\epsilon)$ is convergent then $(\mathcal{B}f)(\epsilon)$ exists and coincides with the usual sum.

More generally, $f(\epsilon)$ is an asymptotic expansion of $(\mathcal{B}f)(\epsilon)$ as $\epsilon \to 0$.

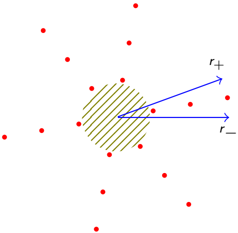
Stokes phenomena

In our examples $\hat{f}(\xi)$ analytically continues to a meromorphic function on $\mathbb{C}$.

We can take the Borel sum along any ray $r = \mathbb{R}_{>0} \cdot \xi_0$ containing no poles of $\hat{f}(\xi)$.



$f(\epsilon) = \sum a_k \epsilon^{k+1}$



$\hat{f}(\xi) = \sum a_k \xi^k / k!$

Different rays $r \subset \mathbb{C}$ give different Borel sums $(\mathcal{B}_r f)(\epsilon)$ in different half-planes.

They all have the same asymptotic expansion as $\epsilon \to 0$.

Example: the Stirling series

The Stirling series

$$f(\epsilon) = \sum_{k\geq 0} \frac{B_{k+2}}{(k+2)(k+1)} \epsilon^{k+1},$$

has zero radius of convergence. The Borel transform is convergent near $\xi = 0$

$$\hat{f}(\xi) = \sum_{k\geq 0} \frac{B_{k+2}}{(k+2)!} \xi^k = \xi^{-2}\left(\frac{\xi}{2}\coth(\xi/2) - 1\right),$$

and extends to a meromorphic function on $\mathbb{C}$.

It has poles at the points $2\pi i m$ for $m \in \mathbb{Z} \setminus \{0\}$, so we choose a ray $r \subset \mathbb{C} \setminus i\mathbb{R}$.

Then $r \subset \pm\{\xi \in \mathbb{C} : \mathrm{Re}(\xi) > 0\}$, and we get one of two Borel sums

$$(\mathcal{B}_r f)(\epsilon) = \pm \log \Upsilon(\pm\epsilon^{-1}), \qquad \Upsilon(w) = \frac{e^w \cdot \Gamma(w)}{\sqrt{2\pi} \cdot w^{w-\frac{1}{2}}}.$$

Topological string free energy of the resolved conifold

We fix $(v,w)\in(\mathbb{C}^*)^2$ with $\mathrm{Im}(v/w)>0$. Set $t=v/w$ and $\lambda=2\pi\epsilon/w$.

The topological string free energy is the Gromov-Witten generating function

$$\mathcal{F}(v,w,\epsilon)=\mathcal{F}(t,\lambda)=\sum_{g\geq 0}\Big(\sum_{d\geq 0}\mathrm{GW}(d\beta,g)e^{2\pi i d t}\Big)\lambda^{2g-2}.$$

The sums over d are convergent, but the sum over g is not. We get a formal series

$$\mathcal{F}(\epsilon)=(\zeta(3)-\mathrm{Li}_3(e^{2\pi i v/w}))\Big(\frac{2\pi i\epsilon}{w}\Big)^{-2}+\frac{1}{12}\mathrm{Li}_1(e^{2\pi i v/w})$$
$$+\sum_{g\geq 2}\frac{B_{2g}\,\mathrm{Li}_{3-2g}(e^{2\pi i v/w})}{2g\,(2g-2)!}\Big(\frac{2\pi i\epsilon}{w}\Big)^{2g-2}+\sum_{g\geq 2}\frac{B_{2g}\,B_{2g-2}}{2g(2g-2)\,(2g-2)!}\Big(\frac{2\pi i\epsilon}{w}\Big)^{2g-2}.$$

Theorem (Pasquetti-Schiappa, Alim-Saha-Teschner-Tulli)

The series $\mathcal{F}(\epsilon)$ is Borel summable along a generic ray $r\subset\mathbb{C}^*$.

More precisely ...

Work of Alim-Saha-Teschner-Tulli shows that:

- The Borel transform $\hat{\mathcal{F}}(\xi)$ extends to a meromorphic function on $\mathbb{C}$.
- The poles lie on the rays spanned by $\pm(v + nw)$ and $\pm w$.
- The series $\mathcal{F}(\epsilon)$ is Borel summable along all other rays $r \subset \mathbb{C}^*$.
- The Borel sum is log of a Barnes triple sine function.
- The Stokes phenomena can be described in terms of DT invariants.

Closely-related work of Garoufalidis-Kashaev on resurgence for the Fadeev dilogarithm.

Link with DT invariants

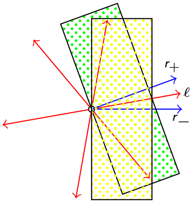
For each non-Stokes ray $r \subset \mathbb{C}^*$ define $X_r \colon \mathbb{H}_r \to \mathbb{T} \cong (\mathbb{C}^*)^4$ by

$$\frac{\partial}{\partial \epsilon} \log X_{r,\delta^\vee}(v,w,\epsilon) = \frac{\partial}{\partial w}\mathcal{F}_r(v,w,\epsilon), \qquad \frac{\partial}{\partial \epsilon} \log X_{r,\beta^\vee}(v,w,\epsilon) = \frac{\partial}{\partial v}\mathcal{F}_r(v,w,\epsilon).$$

$$X_{r,\beta}(\epsilon) = \exp(v/\epsilon), \qquad X_{r,\delta} = \exp(w/\epsilon).$$

Then if $r_\pm$ are small perturbations of a Stokes ray $\ell \subset \mathbb{C}^*$ we have

$$X_{r_+}(\epsilon) = \mathbb{S}(\ell)(X_{r_-}(\epsilon)), \qquad \epsilon \in \mathbb{H}_{r_+} \cap \mathbb{H}_{r_-}$$



Conclusion

Associated to each point of the Kähler moduli space there are:

- a countable collection of rays $\ell = \mathbb{R}_{>0} \cdot e^{i\pi\phi} \subset \mathbb{C}^*$,
- corresponding (partially-defined) Poisson automorphisms $\mathbb{S}_\ell$ of $\mathbb{T} \cong (\mathbb{C}^*)^n$.

This can be obtained in two different ways:

- by considering a stability condition on $\mathcal{D} = \mathcal{D}^b \operatorname{Coh}(X)$ and its DT invariants,
- by studying Borel sums of the GW generating function and their Stokes behaviour.

Does this extend to the whole of stability space? Does it work more generally?

Next case to consider: CY threefolds $u^2 + v^2 + w^2 = q(x)$ / theories of class $S[A_1]$.

3. Non-linear Frobenius structures

What is the geometric setting for all this?

The answer is suggested by an analogy with Frobenius manifolds.

Associated to each point of a semi-simple Frobenius manifold M there are:

- a finite collection of rays $\ell = \mathbb{R}_{>0} \cdot e^{i\pi\phi} \subset \mathbb{C}^*$,
- corresponding Stokes factors $\mathbb{S}_\ell \in \mathrm{GL}(T_{M,m})$.

A Frobenius structure defines a pencil of flat, torsion-free connections on T_M.

Slightly more: there is an extended connection on the pullback of T_M to $M \times \mathbb{P}^1$.

Restricted to $\{m\} \times \mathbb{P}^1$ it takes the form

$$\nabla_m = d - \Big(\frac{U}{\epsilon^2} + \frac{V}{\epsilon}\Big) d\epsilon.$$

The irregular singularity at $\epsilon = 0$ leads to divergent formal solutions, Borel sums etc.

Non-linear version: Joyce structures

Assume that $\langle -, - \rangle$ is non-degenerate so the Poisson torus $\mathbb{T} \cong (\mathbb{C}^*)^n$ is symplectic.

Replace the group $\mathrm{GL}_n(\mathbb{C})$ in the Frobenius story by $\mathrm{Symp}(\mathbb{T})$.

Look for a pencil of non-linear, flat, symplectic connections on T_M.

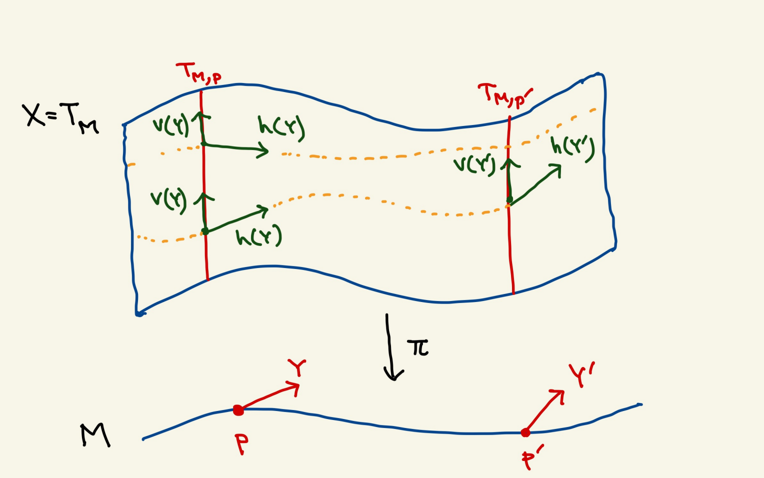
Along with other features, e.g. a $\mathbb{C}^*$-action, this leads to the notion of a Joyce structure.

We expect a Joyce structure on space of stability conditions of CY_3 category.

But we need conditions on the growth rates of the DT invariants.

Constructing the Joyce structure from the DT invariants involves solving
Riemann-Hilbert problems: this is hard!

Lifts of tangent vectors



Pencil of non-linear connections on the tangent bundle

Let M be a complex manifold with tangent bundle $\pi\colon X = T_M \to M$.

There is a canonical isomorphism $\nu\colon \pi^*(T_M) \to \ker(\pi_*)$. Set $v = i \circ \nu$.

Fix a non-linear connection on π, i.e. a splitting $h\colon \pi^*(T_M) \to T_X$.

$$0 \longrightarrow \ker(\pi_*) \xrightarrow{\ i\ } T_X \xrightarrow{\ \pi_*\ } \pi^*(T_M) \longrightarrow 0, \qquad h\colon \pi^*(T_M) \to T_X, \quad \nu\colon \pi^*(T_M) \to \ker(\pi_*)$$

Consider the pencil of connections $h_\epsilon = h + \epsilon^{-1} v$ with $\epsilon^{-1} \in \mathbb{C}$.

Take a holomorphic symplectic form ω on M.

The fibres $\pi^{-1}(m)$ are symplectic manifolds.

Assume that all the connections h_ϵ are flat and symplectic.

Joyce structure and associated hyperkähler structure

For a Joyce structure we impose extra symmetries: invariance under

- a $\mathbb{C}^*$-action on M lifted to X,
- translation by an integral affine structure $T_M^{\mathbb{Z}} \subset T_M$,
- the involution $-1\colon T_M \to T_M$.

We have a splitting $T_X = \mathrm{im}(v) \oplus \mathrm{im}(h) \cong T_M \oplus T_M$.

This gives a complex hyperkähler structure on X:

$$g = \begin{pmatrix} 0 & \omega \\ \omega & 0 \end{pmatrix}, \qquad I = \begin{pmatrix} i \cdot \mathbb{1} & 0 \\ 0 & -i \cdot \mathbb{1} \end{pmatrix}, \qquad J = \begin{pmatrix} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}.$$

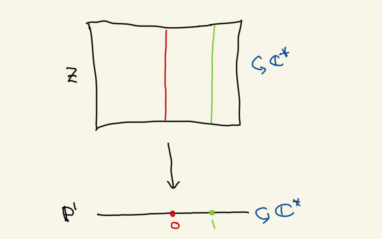
Thus (I, J, K) preserve g and are parallel for ∇^{LC} on T_X.

Twistor space of a Joyce structure

The image of $h_\epsilon = h + \epsilon^{-1} v$ is an integrable distribution $H_\epsilon \subset T_X$.

Define the space of leaves $Z_\epsilon = X/H_\epsilon$.

Varying ϵ gives a twistor space $\pi \colon Z \to \mathbb{P}^1$.



There is a $\mathbb{C}^*$-action on Z lifting the one on $\mathbb{P}^1$.

The central fibre is $Z_0 = M$.

In progress: class $S[A_1]$ case and generating functions

Moduli-theoretic construction of Joyce structures for theories of class $S[A_1]$.

- Partly joint with Nikita Nikolaev and Menelaos Zikidis.
- Like a complexified Hitchin system, but much simpler ("conformal limit").
- Twistor fibres Z_ϵ for $\epsilon \in \mathbb{C}^*$ have an étale map[1] to the cluster variety!

Use the symplectic geometry of the Joyce structure to define generating functions.

- The complex hyperkähler manifold X is the space of twistor lines.
- This gives a symplectic map $F\colon X \to Z_1 \times Z_\infty$.
- Choosing symplectic potentials gives a generating function.
- In the conifold example this reproduces the partition function.
- In the class $S[A_1]$ case of the A_2 quiver we get the Painlevé I τ-function.

[1] I made a mistake in the talk here, by claiming they were *equal* to the cluster variety.